

Chapter 3. Calculus of Variations

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 6)

In preparation for our study of Lagrangian and Hamiltonian dynamics in later chapters, we must first consider the principles and techniques of the calculus of variations.

3.1 Statement of the Problem

Let us first define an integral J such that

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx \quad (3.1)$$

where f is a function of the dependent variable $y(x)$ and its derivative $y'(x) \equiv dy/dx$ (the semicolon in f separates them from the independent variable x). The problem consists of imposing *virtual* variations on $y(x)$ and finding which variation brings the functional integral J to an *extremum*. The limits of integration are kept fixed during this process. In order to keep track of the different versions of $y(x)$ we will extend its definition to include a new parameter α such that

$$y(\alpha, x) = y(x) + \alpha\eta(x) \quad (3.2)$$

where $\eta(x)$ is some function that has a continuous first derivative, and vanishes at the integration limits. We therefore impose the condition that $\eta(x_1) = \eta(x_2) = 0$. We further define the function corresponding to $\alpha = 0$, $y(0, x) = y(x)$, as the one that yields the extremum for J . The situation is shown in Figure 3.1. For example, if $y(x)$ brings J to a minimum, then any other function must make J increase. With the introduction of the parameter α in equation (3.2), the integral also becomes a function of α

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\} dx \quad (3.3)$$

The condition necessary for equation (3.3) to be an extremum is

$$\left. \frac{\partial J}{\partial \alpha} \right|_{\alpha=0} = 0 \quad (3.4)$$

for all function $\eta(x)$.

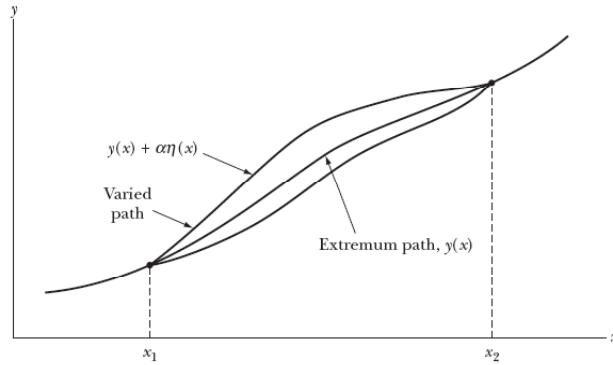


Figure 3.1 – The function $y(x)$ is the path that makes the functional J an extremum. The neighboring functions $y(x) + \alpha\eta(x)$ vanish at the integration limits and may be close to $y(x)$, but they are not extrema.

Example

Consider the function $f = (y')^2$, with $y = x$. Add to $y(x)$ the function $\eta(x) = \sin(x)$, and find the functional $J(\alpha)$ with integration limits 0 and 2π . Show that the extremum of $J(\alpha)$ happens when $\alpha = 0$.

Solution. We first extend $y(x)$ to include α and $\eta(x)$

$$y(\alpha, x) = x + \alpha \sin(x). \quad (3.5)$$

We then calculate

$$\begin{aligned} J(\alpha) &= \int_0^{2\pi} (1 + \alpha \cos(x))^2 dx \\ &= \int_0^{2\pi} (1 + 2\alpha \cos(x) + \alpha^2 \cos^2(x)) dx \\ &= \int_0^{2\pi} \left(1 + \frac{\alpha^2}{2} + 2\alpha \cos(x) + \frac{\alpha^2}{2} \cos(2x) \right) dx \\ &= 2\pi + \alpha^2\pi. \end{aligned} \quad (3.6)$$

We see that $J(\alpha)$ is always greater than $J(0)$. It can also be verified that $\partial J / \partial \alpha|_{\alpha=0} = 0$.

3.2 Euler's Equation

We now calculate the derivative of $J(\alpha)$ (i.e., of equation (3.3)) to meet the condition expressed by equation (3.4)

$$\begin{aligned}\frac{\partial J}{\partial \alpha} &= \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx.\end{aligned}\tag{3.7}$$

Since $y(\alpha, x) = y(x) + \alpha\eta(x)$, we have $\partial y/\partial \alpha = \eta(x)$ and $\partial y'/\partial \alpha = d\eta/dx$. Inserting these two derivatives in equation (3.7)

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \frac{d\eta}{dx} \right) dx.\tag{3.8}$$

We integrate by parts the second term in the integral

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx} dx = \frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) dx.\tag{3.9}$$

The first term on the right hand side vanishes since $\eta(x_1) = \eta(x_2) = 0$. Inserting the remaining term in equation (3.8) we get

$$\begin{aligned}\frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta(x) dx.\end{aligned}\tag{3.10}$$

Even if $\eta(x)$ is subjected to the conditions that it vanishes at the integration limits and that it has a continuous first derivative, it is otherwise completely arbitrary. It must be, therefore, that the quantity that is between the parentheses in the last equation of (3.10) cancels when $\partial J/\partial \alpha|_{\alpha=0} = 0$. That is

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0}\tag{3.11}$$

where y and y' are now the original functions since α is now set to zero. Equation (3.11) is the necessary condition for the functional J to be an extremum, and is known as **Euler's equation**.

Examples

1. *The shortest distance between two points in a plane.* An element of length in a plane is

$$ds = \sqrt{dx^2 + dy^2} \quad (3.12)$$

and the total length of any curve going between point (x_1, y_1) and (x_2, y_2) is

$$I = \int_{(x_1, y_1)}^{(x_2, y_2)} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (3.13)$$

The condition that the curve be the shortest path is that I be a minimum. We, therefore, define

$$f = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \quad (3.14)$$

Inserting f in equation (3.11) with

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}, \quad (3.15)$$

we have

$$\frac{d}{dx} \left(\frac{y'}{\sqrt{1 + (y')^2}} \right) = 0 \quad (3.16)$$

or

$$\frac{y'}{\sqrt{1 + (y')^2}} = \text{cste}. \quad (3.17)$$

This can only be true if

$$y' = a, \tag{3.18}$$

where a is a constant. Integrating equation (3.18) finally yields

$$y = ax + b, \tag{3.19}$$

with b an another constant (of integration). We recognize in equation (3.19) the equation of a straight line.

2. *The problem of the brachistochrone.* Consider a particle constrained to move in a force field (e.g., gravity), starting at rest from a point (x_1, y_1) to some lower point (x_2, y_2) . Find the path that allows the particle to accomplish the transit in the least possible time. We assume that the force on the particle is constant and that there is no friction.

Solution. Let's choose the point (x_1, y_1) as the origin, and assume that the force field is directed along the positive x -axis (see Figure 3.1). Because of the assumptions of a constant force field (i.e., the field is conservative) and the absence of friction, the total energy $T + U$ must be conserved. Furthermore, since the particle starts from rest at (x_1, y_1) and that we chose this point as the origin (i.e., $U = 0$), we have

$$T + U = 0. \tag{3.20}$$

At the arrival point (x_2, y_2) we have $T = mv^2/2$, and $U = -mgx$. Using equation (3.20), the velocity of the particle will, therefore, be

$$v = \sqrt{2gx}. \tag{3.21}$$

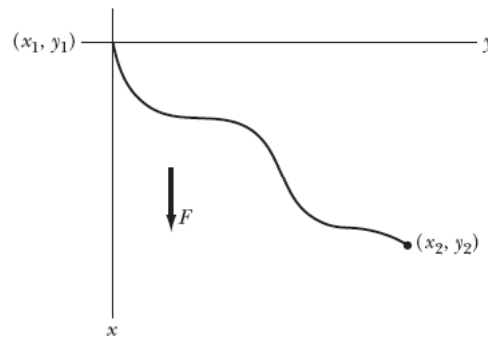


Figure 3.2 – The *brachistochrone* problem. We must find the path that will bring a particle subjected to the force field F from point (x_1, y_1) to point (x_2, y_2) in the least amount of time.

The total time taken by the particle to travel the distance between the two points is given by the following integral

$$\begin{aligned} t &= \int_{(x_1, y_1)}^{(x_2, y_2)} \frac{ds}{v} = \int_{(x_1, y_1)}^{(x_2, y_2)} \sqrt{\frac{dx^2 + dy^2}{2gx}} \\ &= \int_{x_1=0}^{x_2} \sqrt{\frac{1+(y')^2}{2gx}} dx. \end{aligned} \quad (3.22)$$

We can, therefore, identify the function to be used in Euler's equation as

$$f = \sqrt{\frac{1+(y')^2}{2gx}}. \quad (3.23)$$

When we apply Euler's equation (3.11) we find that

$$\frac{\partial f}{\partial y} = 0, \quad (3.24)$$

which in turn implies that

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d}{dx} \left(\frac{y'}{\sqrt{2gx(1+(y')^2)}} \right) = 0. \quad (3.25)$$

This last equation can be written as

$$\frac{y'}{\sqrt{x(1+(y')^2)}} = \frac{1}{\sqrt{2a}}, \quad (3.26)$$

where a is a constant. If now square equation (3.26), rearrange the equation (we multiply by x on both sides of the equality), and rewrite the result in the form of an integral we get

$$y = \int \frac{x dx}{\sqrt{2ax - x^2}}. \quad (3.27)$$

We now make the following change of variable

$$\begin{aligned} x &= a(1 - \cos(\theta)) \\ dx &= a \sin(\theta) d\theta, \end{aligned} \quad (3.28)$$

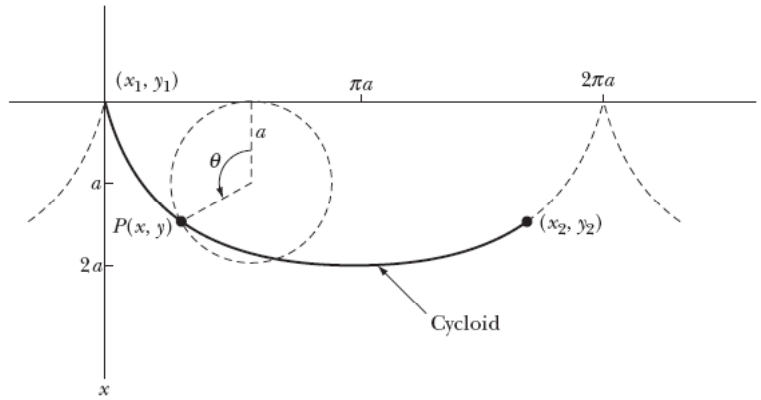


Figure 3.3 – The solution of the brachistochrone problem: a cycloid.

and equation (3.27) becomes

$$y = \int_0^\theta a(1 - \cos(\theta')) d\theta'. \quad (3.29)$$

This last integral is easily solved to give

$$y = a(\theta - \sin(\theta)) + cste. \quad (3.30)$$

Setting the constant in equation (3.30) to zero, we now have the parametric equations of a *cycloid*, that is

$$\begin{aligned} x &= a(1 - \cos(\theta)) \\ y &= a(\theta - \sin(\theta)). \end{aligned} \quad (3.31)$$

The path of the particle described by equation (3.31) is shown in Figure 3.3.

3. *Minimum surface of revolution.* Suppose we form a surface of revolution by taking some curve passing between two fixed end points (x_1, y_1) and (x_2, y_2) defining the (x, y) -plane, and revolving it around the y -axis (see Figure 3.4). The problem is to find the curve for which the surface area is a minimum.

Solution. The area of a strip of surface is

$$dA = 2\pi x dx \sqrt{1 + (y')^2}, \quad (3.32)$$

and that for the total area is

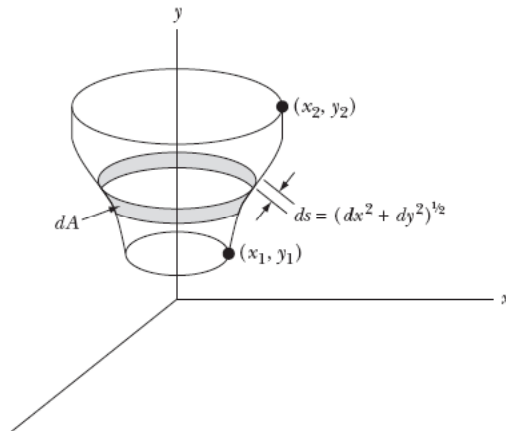


Figure 3.4 – Minimum surface of revolution. The geometry of the problem and area dA are indicated to minimize the surface of revolution around the y -axis.

$$A = 2\pi \int_{x_1}^{x_2} x \sqrt{1 + (y')^2} dx. \quad (3.33)$$

To find the extremum we define

$$f = x \sqrt{1 + (y')^2}, \quad (3.34)$$

and insert it in equation (3.11). We find

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{xy'}{\sqrt{1 + (y')^2}}, \quad (3.35)$$

which implies that

$$\frac{xy'}{\sqrt{1 + (y')^2}} = a, \quad (3.36)$$

with a some constant. Squaring the above equation and factoring terms, we have

$$(y')^2 (x^2 - a^2) = a^2, \quad (3.37)$$

or solving,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \quad (3.38)$$

The general solution of this differential equation is

$$y = a \int \frac{dx}{\sqrt{x^2 - a^2}} = a \cosh^{-1} \left(\frac{x}{a} \right) + b, \quad (3.39)$$

or

$$x = a \cosh \left(\frac{y-b}{a} \right), \quad (3.40)$$

where b is another constant of integration. Equation (3.40) is that of a *catenary*, the curve of a flexible cord hanging freely between two points of support.

3.3 The δ Notation

In analyses that use calculus of variations, or in physics, we often encounter a different notation than what was presented in the preceding sections. The so-called δ notation can be used to rewrite equation (3.10) as

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} d\alpha dx, \quad (3.41)$$

which is then expressed as

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y dx, \quad (3.42)$$

where

$$\begin{aligned} \frac{\partial J}{\partial \alpha} d\alpha &\equiv \delta J \\ \frac{\partial y}{\partial \alpha} d\alpha &\equiv \delta y. \end{aligned} \quad (3.43)$$

The condition of extremum then becomes

$$\delta J = \delta \int_{x_1}^{x_2} f \{y, y'; x\} dx = 0. \quad (3.44)$$

We now solve this equation

$$\begin{aligned}
\delta J &= \int_{x_1}^{x_2} \delta f \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx.
\end{aligned} \tag{3.45}$$

However,

$$\begin{aligned}
\delta y' &= \delta \left(\frac{dy}{dx} \right) = \delta \left(\lim_{\Delta x \rightarrow 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \right) \\
&= \lim_{\Delta x \rightarrow 0} \frac{\delta y(x + \Delta x) - \delta y(x)}{\Delta x} \\
&= \frac{d}{dx}(\delta y).
\end{aligned} \tag{3.46}$$

Inserting this equation in equation (3.45), we get

$$\begin{aligned}
\delta J &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx}(\delta y) \right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y dx,
\end{aligned} \tag{3.47}$$

where we have integrated by parts to derive the last equation. Since δy is arbitrary, δJ can only equal zero if the quantity in parentheses vanishes. This quantity is, as we saw before, Euler's equation.

It is important to realize that the δ notation is only an expedient for the derivation of Euler's equation. The varied path represented by δy is a virtual displacement from the actual path that is consistent with the forces and constraint of the problem, but that is operated with the independent variable x being kept fixed. (The independent variable, when we apply calculus of variations to physical problems, will become the time. In that context, the varied path δy , in fact, does not even need to correspond to a possible path of motion.)

3.4 Functions with Several Dependent Variables

Euler's equation (i.e., equation (3.11)) previously derived is the solution of the variation problem when only one dependent function $y(x)$ was to be found to bring the integral J to an extremum. When dealing with physical problems, it is more common that several such functions are involved. We then have

$$f = f\{y_1(x), y_1'(x), \dots, y_n(x), y_n'(x); x\}, \quad (3.48)$$

or simply

$$f = f\{y_i(x), y_i'(x); x\}, \quad i = 1, 2, \dots, n. \quad (3.49)$$

Just as was done for the case of a single function, we write

$$y_i(\alpha, x) = y_i(x) + \alpha \eta_i(x), \quad (3.50)$$

and

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} \right) \eta_i(x) dx, \quad (3.51)$$

where Einstein's summation convention was used. Because the variations $\eta_i(x)$ are all independent of one another, the vanishing of equation (3.51) when $\alpha = 0$ implies that each equation in parentheses vanishes simultaneously. That is

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0, \quad i = 1, 2, \dots, n.} \quad (3.52)$$

3.5 Euler's Equation with Equations of Constraints

The problem of finding an extremum can be further expanded to include cases where constraints are imposed. For example, determining the shortest path between two points can be complicated by the fact that we may require the path to follow the surface of a given shape (e.g., a sphere). When this is the case, one must take the equations that make these requirements explicit. These are called **equations of constraints**.

Let's consider the case where we have two dependent variables y and z with one equation of constraint, then

$$f = f\{y, y', z, z'; x\}. \quad (3.53)$$

We now write equation (3.51) for the case of two variables

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] dx. \quad (3.54)$$

There is, however, an equation of constraint that we must somehow include in the problem. It is of the form

$$g\{y, z; x\} = 0. \quad (3.55)$$

For example, in the case of restricting the problem to a great circle (of radius a) on the surface of a sphere, we would have

$$y^2 + z^2 - a^2 = 0. \quad (3.56)$$

(Incidentally, constraints that can be written in a form similar to equation (3.55) are called *holonomic*. Other types of constraint that can't be expressed in this fashion (e.g., if “=” is replaced by “≤”) are called *non-holonomic*.)

Because of equation (3.55), the variations $\partial y/\partial\alpha$ and $\partial z/\partial\alpha$ are no longer independent, so the two terms in the parentheses of equation (3.54) do not vanish separately when $\alpha = 0$. If we differentiate the equation of constraint, we have

$$dg = \left(\frac{\partial g}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial \alpha} \right) d\alpha = 0. \quad (3.57)$$

We can evaluate the derivatives with respect to α with

$$\begin{aligned} y(\alpha, x) &= y(x) + \alpha\eta_1(x) \\ z(\alpha, x) &= z(x) + \alpha\eta_2(x), \end{aligned} \quad (3.58)$$

which gives

$$\frac{\partial y}{\partial \alpha} = \eta_1(x), \quad \frac{\partial z}{\partial \alpha} = \eta_2(x). \quad (3.59)$$

Inserting these in equations (3.57), and (3.54), we respectively get

$$\frac{\partial g}{\partial y} \eta_1(x) = -\frac{\partial g}{\partial z} \eta_2(x), \quad (3.60)$$

and

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \eta_2(x) \right] dx. \quad (3.61)$$

We now insert equation (3.60) in equation (3.61) while slightly rearranging the terms

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \left(\frac{\partial g / \partial y}{\partial g / \partial z} \right) \right] \eta_1(x) dx. \quad (3.62)$$

The terms within the brackets in this last equation now include a contribution from the equation of constraint, and can be set to zero. That is

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \left(\frac{\partial g}{\partial z} \right)^{-1}. \quad (3.63)$$

The left-hand side of equation (3.63) involves only derivatives of f and g with respect to y , y' , and x , while the right-hand side involves only derivatives with respect to z , z' and x . But because both y and z are functions of x , the only way that the two sides of equation (3.63) can be equal for all value of x is if they are both equal to the same function of x . We define this function as $-\lambda(x)$, and we have

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda(x) \frac{\partial g}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda(x) \frac{\partial g}{\partial z} &= 0 \end{aligned} \quad (3.64)$$

We, therefore, need to find a solution to three equations (i.e., $y(x)$, $z(x)$, and $\lambda(x)$) in order to completely solve the problem. For this, the two equations (3.64) along with the equation of constraint (3.55), for a total of three, will be used. The function $\lambda(x)$ is known as a **Lagrange undetermined multiplier**.

The procedure described in this section can readily be generalized to the case of several dependent variables and more than a single equation of constraints are involved. We then have the following solution

$$\boxed{\begin{aligned} \frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} + \lambda_j(x) \frac{\partial g_j}{\partial y_i} &= 0 \\ g_j \{y_i; x\} &= 0. \end{aligned}} \quad (3.65)$$

It is to be noted that the equation of constraints will often be presented in a differential form such as

$$\frac{\partial g_j}{\partial y_i} dy_i = 0. \quad (3.66)$$

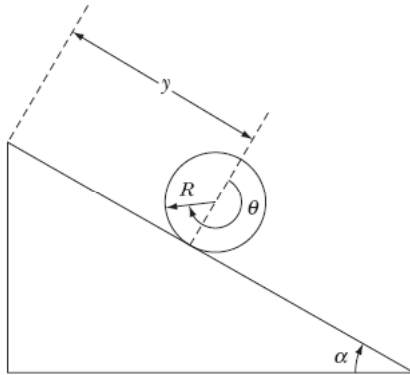


Figure 3.5 – A disk rolling on an incline plane without slipping.

Example

Determine the equation of constraint for a disk rolling on an incline plane without slipping.

Solution.

From Figure 3.5 we can easily determine the relation the two coordinates y and θ as

$$y = R\theta, \quad (3.67)$$

with R the radius of the disk. The equation of constraint is then

$$g(y, \theta) = y - R\theta = 0, \quad (3.68)$$

or in a differential form

$$\frac{\partial g}{\partial y} = 1, \quad \frac{\partial g}{\partial \theta} = -R. \quad (3.69)$$

3.6 Equation of Constraints as Integrals

It is also possible that an equation of constraints can be stated in the form of an integral. For example, suppose that we want to find an extremum for a functional integral J by determining the corresponding curve which satisfies the boundary conditions $y(a) = A$ and $y(b) = B$

$$J[y] = \int_a^b f\{y, y'; x\} dx. \quad (3.70)$$

Suppose also that the constraint is stated as a functional (or integral) K that fixes the value for the length of the curve

$$K[y] = \int_a^b g\{y, y'; x\} dx = \ell, \quad (3.71)$$

where ℓ is the aforementioned length of y . Similarly as was done before, we now consider virtual variations on the curve such that $y(\alpha, x) = y(x) + \alpha\eta(x)$. Since these variations apply to both J and K , we can write

$$\begin{aligned} J &= J(\alpha) \\ K &= K(\alpha), \end{aligned}$$

and,

$$\begin{aligned} \left. \frac{\partial J}{\partial \alpha} \right|_{\alpha=0} &= 0 \\ \left. \frac{\partial K}{\partial \alpha} \right|_{\alpha=0} &= 0. \end{aligned} \quad (3.72)$$

Furthermore, since K , and $\partial K/\partial \alpha$ are not a function of x , we can introduce a Lagrange undetermined multiplier such that

$$\left(\frac{\partial J}{\partial \alpha} + \lambda \frac{\partial K}{\partial \alpha} \right)_{\alpha=0} = \frac{\partial}{\partial \alpha} (J + \lambda K)_{\alpha=0} = 0. \quad (3.73)$$

Take note that in this case λ is truly a constant; it is not a function of the independent variable x . We now rewrite equation (3.73) as

$$\frac{\partial}{\partial \alpha} \int_a^b (f + \lambda g) dx \Big|_{\alpha=0} = 0. \quad (3.74)$$

We can find the extremum for equation (3.74) in the same way as was done before, and we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda \left(\frac{\partial g}{\partial y} - \frac{d}{dx} \frac{\partial g}{\partial y'} \right) = 0, \quad (3.75)$$

with the constraints $y(a) = A$, $y(b) = B$, and $K[y] = \ell$.

Example

We want to maximize the area of a closed curve $y(x)$ that has a length ℓ .

Solution. The area inside a closed curve and its length are

$$\begin{aligned} J &= \iint dx dy \\ K &= \oint ds = \ell. \end{aligned} \tag{3.76}$$

Since the curve is closed, it is appropriate to change coordinates and consider the curve $r(\theta)$ instead of $y(x)$, with θ as the independent variable instead of x . Equations (3.76) can be rewritten as

$$\begin{aligned} J &= \int_0^{2\pi} \int_0^{r(\theta)} r' dr' d\theta = \frac{1}{2} \int_0^{2\pi} r^2(\theta) d\theta \\ K &= \int_0^{2\pi} r(\theta) d\theta = \ell. \end{aligned} \tag{3.77}$$

We can now identify the two functions f and g

$$f = \frac{1}{2} r^2(\theta), \quad g = r(\theta). \tag{3.78}$$

Using equation (3.75) we find

$$r(\theta) + \lambda = 0, \tag{3.79}$$

which implies that r is a constant. Using equation (3.77) we find

$$\begin{aligned} r &= -\lambda = \frac{\ell}{2\pi} \\ J &= \pi r^2 = \frac{\ell^2}{4\pi}. \end{aligned} \tag{3.80}$$

We therefore find that the area is maximized if the curve is that of a circle.